

ON TOPOLOGICAL COMPLEXITY AND LS-CATEGORY

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ABSTRACT. We present some results supporting the Iwase-Sakai conjecture about coincidence of the topological complexity $TC(X)$ and monoidal topological complexity $TC^M(X)$. Using these results we provide lower and upper bounds for the topological complexity of the wedge $X \vee Y$. We use these bounds to give a counterexample to the conjecture asserting that $TC(X') \leq TC(X)$ for any covering map $p : X' \rightarrow X$.

We discuss a possible reduction of the monoidal topological complexity to the LS-category. Also we apply the LS-category to give a short proof of the Arnold-Kuiper theorem.

1. INTRODUCTION

Let $PX = X^{[0,1]}$ denote the space of all paths in X . Let $i_X : X \rightarrow PX$ be the inclusion of X into PX as a subspace of constant paths. There is a natural fibration $\pi : PX \rightarrow X \times X$ defined as $\pi(f) = (f(0), f(1))$ for $f \in PX$, $f : [0, 1] \rightarrow X$.

Let X be an ENR. A section $s : X \times X \rightarrow PX$ of π is called a *motion planning algorithm*. We say that a motion planning algorithm s has *complexity* k if $X \times X$ can be presented as a disjoint union $F_1 \cup \dots \cup F_k$ of ENRs such that s is continuous on each F_i . The *topological complexity* $TC(X)$ of a space X was defined by Farber as the minimum of k such that there is a motion planning algorithm of complexity k [F1]. Equivalently, $TC(X)$ is the minimal number k such that $X \times X$ admits an open cover U_1, \dots, U_k such that over each U_i there is a continuous section of π .

We say that a motion planning algorithm $s : X \times X \rightarrow PX$ is *reserved* if $s|_{\Delta X} = i_X$ where $\Delta X \subset X \times X$ is the diagonal. In other words, if the initial position of a robot in the configuration space X coincides with the terminal position, then the algorithm keeps the robot still. This

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condition on the motion planning algorithms seems to be very natural. The corresponding complexity of a space X was denoted by Iwase and Sakai as $TC^M(X)$ and was called the *monoidal topological complexity* of X [IS1]. In the original definition they additionally assumed that all sets U_i contain the diagonal. Their definition agrees with the above since their condition always can be achieved by reduction of an open cover U_1, \dots, U_k with reserved sections s_i to a closed cover F_1, \dots, F_k , $F_i \subset U_i$, then by adding the diagonal to each F_i with the natural extension of the sections \bar{s}_i , and then by taking open enlargement V_i of the sets $F_i \cup \Delta X$ that admit extensions of the sections \bar{s}_i .

Iwase and Saki conjectured that $TC^M(X) = TC(X)$. In fact, first they gave a proof to the conjecture in [IS1] and then withdrew it in [IS2]. We prove this conjecture under the assumption $TC(X) > \dim X + 1$. Also, using the Weinberger Lemma from [F3] we show that the conjecture holds true when X is a Lie group.

The topological complexity is closely related to the Lusternik-Schnirelmann category $\text{cat}(X)$ of a space which is defined as the minimal number k such that X can be covered by k open sets U_1, \dots, U_k all contractible to a point in X . We denote by

$$\text{Cat}(X) = \text{cat}(X) - 1,$$

the reduced LS-category. The reduced category appears naturally in several inequalities in the theory [CLOT]:

$$\text{cup-length}(X) \leq \text{Cat}(X) \leq \dim(X)$$

and

$$\text{Cat}(X \times Y) \leq \text{Cat}(X) + \text{Cat}(Y).$$

In the first inequality the cup-length is taken for any reduced cohomology (possibly twisted).

Some of the formulas for cat translate to similar statements for TC . For example for TC there is an inequality similar to the above for the product of two spaces [F4]. Also there are analogous estimates of TC in terms of the cup product and dimension [F4]. On the other hand, the simple cat formula for the wedge $\text{cat}(X \vee Y) = \max\{\text{cat } X, \text{cat } Y\}$ does not hold for TC . So far there is no nice analog of it for TC . The best that we can prove here is Theorem 3.6 from this paper. Another example is the formula $\text{cat}(Y) \geq \text{cat}(X)$ for a covering map $p : X \rightarrow Y$ which supports an intuitive idea that a covering space is always simpler than the base. So it was natural to assume that the same holds true for TC . I've learned about this problem from Yuli Rudyak. In this paper Theorem 3.8 gives a negative answer to this question.

There have been several attempts to reformulate the topological complexity of X as some modified category of a related space. In this paper we discuss a possible characterization of the monoidal topological complexity in terms of the category. We define a *rel* ∞ category $\infty\text{-cat}(Y)$ of non-compact spaces Y and discuss the problem of coincidence between $\text{cat}(X/A)$ and $\infty\text{-cat}(X \setminus A)$ for a subcomplex $A \subset X$ of a finite complex X . Then we show that $TC^M(X)$ is always between $\text{cat}(X \times X)/\Delta(X)$ and $\infty\text{-cat}(X \times X \setminus \Delta X)$.

Note that both $\text{cat}(X)$ and $TC(X)$ are partial case of the Schwarz genus [Sch]: $\text{cat}(X) = sg(\pi_0 : P_0X \rightarrow X)$ and $TC(X) = sg(\pi : PX \rightarrow X \times X)$ where $P_0X \subset PX$ is the subspace of paths $f : [0, 1] \rightarrow X$ that start in a base point $x_0 \in X$, $f(0) = x_0$, and $\pi_0(f) = f(1)$. We recall the *Schwarz genus* [Sch] of a fibration $p : X \rightarrow Y$ is the minimal number of open sets U_1, \dots, U_k that cover Y and admit sections $s_i : U_i \rightarrow X$ of p . In the paper we estimate the Schwarz genus [Sch] of arbitrary fibration $p : X \rightarrow Y$ in terms the category of its mapping cone C_p .

Finally, we apply the LS-category to give a short proof of the Arnold-Kuiper theorem which states that the orbit space of the action of \mathbb{Z}_2 on the complex projective plane $\mathbb{C}P^2$ by the conjugation is the 4-sphere. Note that this theorem was discovered by Arnold [Ar1] who published his proof much later [Ar2]. It was proven independently by Kuiper [K] and by Massey [M].

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2. MONOIDAL TOPOLOGICAL COMPLEXITY

2.1. Theorem. *For ENR spaces,*

$$TC(X) \leq TC^M(X) \leq TC(X) + 1.$$

This theorem was proved in [IS2]. Since the proof there is too technical we give an alternative proof.

Proof. The first inequality is obvious. Since X is ANR, there is an open neighborhood W of the diagonal ΔX in $X \times X$ and a continuous map $\phi : W \times [0, 1] \rightarrow X$ such that $\phi(x, x', 0) = x$, $\phi(x, x', 1) = x'$, and $\phi(x, x, t) = x$ for all $t \in [0, 1]$, $x, x' \in X$. Let U_1, \dots, U_n be an open cover of $X \times X$ by sets that admit sections $s_i : U_i \rightarrow PX$ of π . Let F be a closed neighborhood of ΔX that lies in W . Then all sets in the open cover $U_1 \setminus F, \dots, U_n \setminus F, W$ of $X \times X$ admit reserved sections. Hence $TC^M(X) \leq n + 1$. \square

Note that the path fibration $\pi : PX \rightarrow X \times X$ restricted over the diagonal defines the free loop fibration $p : LX \rightarrow X$. A *canonical*

section $\bar{s} : \Delta X \rightarrow LX$ of p is defined as $\bar{s}(x) = c_x$, where $c_x : I \rightarrow X$ is the constant map to x .

We use the standard convention to denote the elements of the iterated join product $X_1 * X_2 * \cdots * X_n$ as formal linear combinations $t_1x_1 + t_2x_2 + \cdots + t_nx_n$, $\sum t_i = 1$, $t_i \geq 0$, $x_i \in X_i$ where all summands of the type $0x_i$ are dropped. We use the notation $*^n X$ for the iterated join product of n copies of X with itself.

We recall that a fiber-wise join of maps $f_i : X_i \rightarrow Y$, $i = 1, \dots, n$ is the map

$$f_1 \tilde{*} \cdots \tilde{*} f_n : X_1 \tilde{*}_Y \cdots \tilde{*}_Y X_n \rightarrow Y$$

where

$$X_1 \tilde{*}_Y \cdots \tilde{*}_Y X_n = \{t_1x_1 + \cdots + t_nx_n \in X_1 * \cdots * X_n \mid f_1(x_1) = \cdots = f_n(x_n)\}$$

is the fiber-wise join of spaces X_1, \dots, X_n and

$$(f_1 \tilde{*} \cdots \tilde{*} f_n)(t_1x_1 + \cdots + t_nx_n) = f_i(x_i).$$

Thus, the preimage $(f_1 \tilde{*} \cdots \tilde{*} f_n)^{-1}(y)$ of a point $y \in Y$ is the join product of the preimages $f_1^{-1}(y) * \cdots * f_n^{-1}(y)$.

We define $P_n X = PX \tilde{*}_{X \times X} \cdots \tilde{*}_{X \times X} PX$ and

$$\pi_n = \pi \tilde{*} \cdots \tilde{*} \pi : P_n X \rightarrow X \times X$$

to be the fiber-wise join product of n copies of π . Note that there are imbeddings $P_1 X \subset P_2 X \subset \cdots \subset P_n X$ such that $\pi_i|_{P_{i-1}} = \pi_{i-1}$. Then the section $\bar{s} : X \times X \rightarrow P_1 X$ of π_1 can be regarded as a section of π_n . Also we define $p_1 = p : LX \rightarrow X$, $L_n X = L_{n-1} \tilde{*}_X LX$, and $p_n = p_{n-1} \tilde{*} p : L_n X \rightarrow X$. Note that $\pi_n^{-1}(\Delta X) \cong L_n X$ and p_n is the restriction of π_n to $\pi_n^{-1}(\Delta X)$. Note also that the canonical section \bar{s} defines a trivial subbundle $p'_n : E \rightarrow X$ of p_n with the fiber the $(n-1)$ -simplex Δ^{n-1} .

We recall that a map $p : E \rightarrow B$ satisfies the *Homotopy Lifting Property for a pair* (X, A) if for any homotopy $H : X \times I \rightarrow B$ with a lift $H' : A \times I \rightarrow E$ of the restriction $H|_{A \times I}$ and a lift H_0 of $H|_{X \times 0}$ which agrees with H' , there is a lift $\tilde{H} : X \times I \rightarrow E$ of H which agrees with H_0 and H' . The following is well-known [H]:

2.2. Theorem. *Any Hurwicz fibration $p : E \rightarrow B$ satisfies the Homotopy Lifting Property for CW complex pairs (X, A) .*

2.3. Corollary. *Let $p : E \rightarrow X$ be a Hurewicz fibration with a section $s : X \rightarrow E$. A fiber-wise homotopy $G : A \times I \rightarrow E$ of the restriction $s|_A$ to a closed subset $A \subset X$ can be extended to a fiber-wise homotopy $\bar{G} : X \times I \rightarrow E$ of s provided (X, A) is a CW complex pair.*

2.4. Proposition. *For CW complexes X ,*

(1) $TC(M) \leq n \Leftrightarrow \pi_n : P_n X \rightarrow X \times X$ *admits a section.*

(2) $TC^M(M) \leq n \Leftrightarrow \pi_n : P_n X \rightarrow X \times X$ *admits a section s which agrees with the canonical section over the diagonal $s|_{\Delta X} = \bar{s}$.*

Proof. The statement (1) is a part of a general theorem proven by Schwartz [Sch] for fibrations $q : X \rightarrow Y$: $sg(q) \leq n$ if and only if the n -fold iterated fiber-wise join product $\tilde{*}^n q : \tilde{*}_Y^n X \rightarrow Y$ admits a section.

The implication \Leftarrow in (2) is obvious. For the other direction we note that n reserved sections $s_i : U_i \rightarrow PX$ defined for an open cover U_1, \dots, U_n of $X \times X$ define a section s of π_n with the image $s(\Delta X)$ lying in E . Therefore over ΔX it could be fiber-wise deformed to \bar{s} . By Proposition 2.2 that deformation can be extended to a fiber-wise deformation of s . \square

2.5. Theorem. *The equality*

$$TC(X) = TC^M(X)$$

holds true for k -connected simplicial complexes X such that

$$(k+1)TC(X) > \dim(X) + 1.$$

Proof. Let $TC(X) = n$. Note that the fiber $\pi^{-1}(x, x')$ is homotopy equivalent to the loop space $\Omega(X)$. Since $\Omega(X)$ is $(k-1)$ -connected, the iterated join product $*^n \Omega(X)$ is $((k+1)n-2)$ -connected. We show that any section $s : \Delta X \rightarrow L_n X$ can be fiber-wise joined by a homotopy with a canonical section $\bar{s} : \Delta X \rightarrow L_n X$. By induction on i we construct a section $s_i : X \rightarrow L_n X$, that coincides with \bar{s} on the i -skeleton $X^{(i)}$, together with a fiber-wise homotopy joining s and s_i . Here we use the identification $\Delta X = X$. For $i = 0$ we take paths in the fibers $p_n^{-1}(v)$ joining $s(v)$ and $\bar{s}(v)$ for all $v \in X^{(0)}$. Then we extend them to a fiber-wise homotopy of s to a section s_0 . Assume that s_{i-1} is already constructed and $i \leq \dim X \leq (k+1)n-2$. Independently for every i -simplex $\sigma \subset X$ we consider the problem of joining s_{i-1} with \bar{s} over σ by a fiber-wise homotopy. Since the fiber bundle p_n is trivial over σ with a i -connected fiber, the identity homotopy on the boundary $\partial\sigma$ can be extended to a homotopy between $\bar{s}|_\sigma$ and $s_{i-1}|_\sigma$. This extension can be deformed to a fiber-wise homotopy. All these homotopies together define a fiber-wise homotopy between s_{i-1} and \bar{s} over $X^{(i)}$. Since $(X, X^{(i)})$ is a CW pair, by Proposition 2.2 we can extend it to a fiber-wise homotopy over X .

Let $s : X \times X \rightarrow P_n X$ be a section. On ΔX it can be deformed to a canonical section \bar{s} . Since $(X \times X, \Delta X)$ is a CW pair, by Proposition 2.2 there is a fiber-wise homotopy of s to a section s' that coincides with \bar{s} on ΔX . Therefore, $TC^M(X) \leq n$. \square

2.6. Corollary. $TC(S^m) = TC^M(S^m)$ for all m .

The following is an extension of Weinberger's Lemma from [F3] to the case of monoidal topological complexity.

2.7. Lemma. For a connected Lie group G ,

$$TC(G) = TC^M(G) = \text{cat}(G).$$

Proof. In view of what is already known [F3], it suffices to show the inequality $TC^M(G) \leq \text{cat}(G)$. Let $\text{cat}(G) = n$ and let U_1, \dots, U_n be an open cover of G together with homotopies $H_i : U_i \times [0, 1] \rightarrow G$ contracting U_i to the unit $e \in G$. Clearly, we may assume that $e \notin U_i$ for $i > 1$. Since the inclusion $e \in G$ is a cofibration, we may assume that $H_1(e, t) = e$ for all t . Then for the open cover of $G \times G$ as defined in [F3]

$$W_i = \{(a, b) \in G \times G \mid a^{-1}b \in U_i\}$$

the sections $s_i : W_i \rightarrow PG$ defined as

$$s_i(a, b)(t) = ah_i(a^{-1}b, t) \in G, \quad (a, b) \in W_i$$

are reserved. Indeed, $\Delta G \cap W_i = \emptyset$ for $i > 1$ and

$$s_1(a, a)(t) = ah_1(a^{-1}a, t) = ah_1(e, t) = ae = a$$

for all $(a, a) \in \Delta G$. □

3. TOPOLOGICAL COMPLEXITY OF WEDGE AND COVERING MAPS

A *deformation* of $U \subset Z$ in Z to a subset $A \subset Z$ is a continuous map $D : U \times I \rightarrow Z$ such that: $D(u, 0) = u$, $D(u, 1) \in A$ for all $u \in U$. A *strict deformation* of $U \subset Z$ in Z to $A \subset Z$ is a deformation $D : U \times I \rightarrow Z$ such that $D(u, t) = u$ for all $t \in I$ whenever $u \in A$.

3.1. Proposition. Let X be a metric space. For an open set $U \subset X \times X$ the following are equivalent:

(1) There is a reserved section $s : U \rightarrow PX$ over U of the fibration $\pi : PX \rightarrow X \times X$.

(2) There is a strict deformation $D : U \times I \rightarrow X \times X$ to the diagonal $\Delta X = \{(x, x) \in X \times X \mid x \in X\}$

(3) For any choice of a base point $x_0 \in X$ there is a strict deformation D of U to ΔX which preserves faces $X \times x_0$ and $x_0 \times X$, i.e., for all $t \in I$,

$$D((x, x_0), t) \in X \times x_0 \quad \text{and} \quad D((x_0, x), t) \in x_0 \times X.$$

Proof. (1) \Rightarrow (3). Let $\|x\| = d(x, x_0)$. We define

$$D((x, y), t) = (s(x, y)(\frac{\|x\|}{\|x\| + \|y\|}t), s(x, y)(1 - \frac{\|y\|}{\|x\| + \|y\|}t))$$

if $(x, y) \neq (x_0, x_0)$ and define $D((x_0, x_0), t) = (x_0, x_0)$. Since $s(x, y)(0) = x$ and $s(x, y)(1) = y$, we obtain that $D((x, y), 0) = (x, y)$. Note that

$$D((x, y), 1) = (s(x, y)(\frac{\|x\|}{\|x\| + \|y\|}), s(x, y)(\frac{\|x\|}{\|x\| + \|y\|})) \in \Delta X.$$

Since the section s is reserved, $D((x, x), t) = (s(x, x)(t/2), s(x, x)(t/2)) = (x, x)$. Note that

$$D((x, x_0), t) = (s(x, x_0)(t), s(x, x_0)(1)) = (s(x, x_0)(t), x_0) \in X \times x_0$$

and

$$D((x_0, y), t) = (s(x_0, y)(0), s(x_0, y)(1-t)) = (x_0, s(x_0, y)(1-t)) \in x_0 \times X.$$

The deformation D is continuous at (x_0, x_0) (if defined) since the section $s(x_0, x_0)$ is stationary at (x_0, x_0) .

(3) \Rightarrow (2) is obvious.

(2) \Rightarrow (1). Let $pr_1 : X \times X \rightarrow X$ denote the projection to the first factor and $pr_2 : X \times X \rightarrow X$ to the second. Given a strict deformation D we define a section $s : U \times I \rightarrow PX$ as follows:

$$s(x, y)(t) = \begin{cases} pr_1 D((x, y), 2t) & \text{if } t \leq 1/2 \\ pr_2 D((x, y), 2 - 2t) & \text{if } t \geq 1/2. \end{cases}$$

This path is well-defined since $D((x, y), 1) \in \Delta X$. Clearly it is a path from x to y . If $x = y$, the path is stationary. Thus s is a reserved section. \square

3.2. Proposition. *Let A be a retract of an ENR space X . Then $TC(X) \geq TC(A)$.*

Proof. Let $r : X \rightarrow A$ be a retraction. Let $TC(X) = k$ and let $X \times X = U_1 \cup \dots \cup U_k$ be an open cover together with continuous sections $s_i : U_i \rightarrow PX$. We define sections $\sigma_i : U_i \cap (A \times A) \rightarrow PA$ by the formula $\sigma_i(a_1, a_2)(t) = r(s_i(a_1, a_2)(t))$. \square

We recall that a family \mathcal{U} of subsets of X is called a k -cover, $k \in \mathbb{N}$ if every subfamily that consists of k elements forms a cover of X . We use the following theorem [Dr1].

3.3. Theorem. *Let $\{U'_0, \dots, U'_n\}$ be an open cover of a normal topological space X . Then for any $m = n, n+1, \dots, \infty$ there is an open $(n+1)$ -cover of X , $\{U_k\}_{k=0}^m$ such that $U_k = U'_k$ for $k \leq n$ and $U_k = \cup_{i=0}^n V_i$ is a disjoint union with $V_i \subset U_i$ for $k > n$.*

3.4. Corollary. *Suppose that all sets U'_i , $i = 0, \dots, n$, in the theorem are (strictly) deformable in X to a subspace $A \subset X$. Then the sets U_k for all k are (strictly) deformable in X to A .*

The following proposition is well-known. The trick presented there can be traced back to the work of Kolmogorov on 13th Hilbert's problem [Os].

3.5. Proposition. *Let U_0, \dots, U_{n+m} be an $(n+1)$ -cover of X and let V_0, \dots, V_{m+n} be an $(m+1)$ -cover of Y . Then the sets $W_k = U_k \times V_k$, $k = 0, \dots, n+m$, cover $X \times Y$.*

Proof. Let $(x, y) \in X \times Y$. A point x is covered at least by $m+1$ elements. Otherwise $n+1$ elements that do not cover x would not form a cover of X . That would give a contradiction with the assumption that U_0, \dots, U_{n+m} is an $(n+1)$ -cover of X . Let $x \in U_{i_0} \cap \dots \cap U_{i_m}$. By the assumption, the family V_{i_0}, \dots, V_{i_m} covers Y . Hence $y \in V_{i_s}$ for some s . Then $(x, y) \in W_{i_s}$. \square

3.6. Theorem. *For all ENR spaces X and Y ,*

$$\begin{aligned} \max\{TC(X), TC(Y), cat(X \times Y)\} &\leq TC(X \vee Y) \leq \\ &\leq TC^M(X \vee Y) \leq TC^M(X) + TC^M(Y) - 1 \end{aligned}$$

Proof. Note that $TC(X \vee Y) \geq TC(X), TC(Y)$ by Proposition 3.2. Let $r_X : X \vee Y \rightarrow X$ and $r_Y : X \vee Y \rightarrow Y$ be the retraction collapsing the wedge onto X and Y respectively. The subset

$$X \times Y \subset (X \vee Y) \times (X \vee Y)$$

is covered by $\leq TC(X \vee Y)$ open sets U supplied with a homotopy

$$H_U : U \times I \rightarrow X \vee Y$$

such that $H(x, y, 0) = x$ and $H(x, y, 1) = y$. For each U we define a homotopy $G : U \times I \rightarrow X \times Y$ by the formula

$$G(x, y, t) = (r_X H_U(x, y, t), r_Y H_U(x, y, 1 - t)).$$

Then

$$G(x, y, 0) = (r_X H_U(x, y, 0), r_Y H_U(x, y, 1)) = (r_X(x), r_Y(y)) = (x, y)$$

and

$$G(x, y, 1) = (r_X H_U(x, y, 1), r_Y H_U(x, y, 0)) = (r_X(y), r_Y(x)) = (v_0, v_0)$$

where v_0 is the wedge point in $X \vee Y$. Thus, G contracts U to a point in $X \times Y$.

Let $TC^M(X) = n+1$ and $TC^M(Y) = m+1$. Then there is an open cover $\tilde{U}_0, \dots, \tilde{U}_n$ of $X \times X$ with reserved sections $s_i : \tilde{U}_i \rightarrow PX$,

$i = 0, \dots, n$. Similarly, let $\tilde{V}_0, \dots, \tilde{V}_m$ be an open covering of $Y \times Y$ with reserved sections $\sigma_j : \tilde{V}_j \rightarrow PY$, $j = 0, \dots, m$. By Proposition 3.1 all these sets are strictly deformable to the diagonal in $X \times X$ and $Y \times Y$ respectively. By Corollary 3.4 there is an open $(n+1)$ -cover $\tilde{U}_0, \dots, \tilde{U}_n, \dots, \tilde{U}_{n+m}$ of $X \times X$ by sets strictly deformable to the diagonal. By Proposition 3.1 there are strict deformations

$$D_X^k : \tilde{U}_k \times I \rightarrow X \times X$$

of \tilde{U}_k to ΔX that preserves faces $X \times v_0$ and $v_0 \times X$. Similarly, there is an open $(m+1)$ -cover $\tilde{V}_0, \dots, \tilde{V}_m, \dots, \tilde{V}_{m+n}$ of $Y \times Y$ and there are strict deformations D_Y^k of \tilde{V}_k in $Y \times Y$ to the diagonal ΔY that preserves faces.

We use notations

$$U_k = \tilde{U}_k \cap (X \times v_0) \quad \text{and} \quad V_k = \tilde{V}_k \cap (v_0 \times Y), \quad k = 0, \dots, m+n.$$

Note that U_0, \dots, U_{m+n} is an $(n+1)$ -cover of $X \times v_0 = X$ and V_0, \dots, V_{m+n} is an $(m+1)$ -cover of $v_0 \times Y = Y$. Let $W_k = U_k \times V_k$. By Proposition 3.5 W_0, \dots, W_{m+n} is an open cover of $X \times Y$.

The deformations D_X^k define the deformations $H_k : U_k \times I \rightarrow X \times v_0$ to the point $v_0 \in X$ and the deformations D_Y^k define the deformations $G_k : V_k \times I \rightarrow v_0 \times Y$ to the point $v_0 \in Y$. These deformations define the deformations

$$T_k : W_k \times I \rightarrow X \times Y$$

to the point (v_0, v_0) such that if $W_k \cap (X \times v_0) \neq \emptyset$ then $W_k \cap (X \times v_0) = U_k$ and $T_k|_{U_k \times I} = H_k$ and if $W_k \cap (v_0 \times Y) \neq \emptyset$ then $W_k \cap (v_0 \times Y) = V_k$ and $T_k|_{V_k \times I} = G_k$ for $k = 0, \dots, m+n$.

Symmetrically, define

$$U'_k = \tilde{U}_k \cap (v_0 \times X) \quad \text{and} \quad V'_k = \tilde{V}_k \cap (Y \times v_0), \quad k = 0, \dots, m+n,$$

and corresponding deformations

$$H'_k : U'_k \times I \rightarrow X \quad \text{and} \quad G'_k : V'_k \times I \rightarrow Y$$

to the base points. Define $W'_k = U'_k \times V'_k$. By Proposition 3.5, the family W'_0, \dots, W'_{n+m} is an open cover of $Y \times X$. As before there are deformations

$$T'_k : W'_k \times I \rightarrow Y \times X$$

to the point (v_0, v_0) such that if $W'_k \cap (v_0 \times X) \neq \emptyset$, then $W'_k \cap (v_0 \times X) = U'_k$ and $T'_k|_{U'_k \times I} = H'_k$ and if $W'_k \cap (Y \times v_0) \neq \emptyset$, then $W'_k \cap (Y \times v_0) = V'_k$, $T'_k|_{V'_k \times I} = G'_k$ for $k = 0, \dots, m+n$.

We define open sets

$$O_k = W_k \cup W'_k \cup \tilde{U}_k \cup \tilde{V}_k \subset (X \vee Y) \times (X \vee Y), \quad k = 0, \dots, n+m$$

and note that $\mathcal{O} = \{O_k\}$ covers $(X \vee Y) \times (X \vee Y)$. Note that the set

$$C = (X \vee Y) \times v_0 \bigcup v_0 \times (X \vee Y)$$

defines a partition of $(X \vee Y) \times (X \vee Y)$ in four pieces $X \times X$, $X \times Y$, $Y \times X$, and $Y \times Y$. Also note that the intersection $O_k \cap C \subset U_k \cup V_k \cup U'_k \cup V'_k$. By the construction the deformations D_X^k , D_Y^k , T_k , and T_k all agrees on $O_k \cap C$. Therefore the union of deformations

$$T_k \cup T'_k \cup D_X^k \cup D_Y^k : O_k \times I \rightarrow (X \vee Y) \times (X \vee Y)$$

is a well defined deformation Q_k of O_k to the diagonal $\Delta(X \vee Y)$. Note that for all k , Q_k are strict deformations. By Proposition 3.1 each Q_k defines a reserved section $\alpha_k : O_k \rightarrow P(X \vee Y)$. Therefore,

$$TX^M(X \vee Y) \leq n + m + 1 = TC(X) + T(Y) - 1.$$

□

3.7. Remark. A stronger version of the upper bound of Theorem 3.6 was proposed in [F2], (Theorem 19.1):

$$TC(X \vee Y) \leq \max\{TC(X), TC(Y), \text{cat}(X) + \text{cat}(Y) - 1\}.$$

Since the proof in [F2] contains a gap, we call this inequality *Farber's Conjecture*. Note that Farber's inequality in view of Theorem 3.6 would turns into the equality for spaces X and Y with $\text{Cat}(X \times Y) = \text{Cat}(X) + \text{Cat}(Y)$.

3.8. Theorem. (1) *There is a 2-to-1 covering map $p : E \rightarrow B$ with $TC(E) > TC(B)$.*

(2) *There is a finite complex X with $TC(X) < TC(\tilde{X})$ where \tilde{X} is the universal covering of X .*

Proof. (1) We take $B = T \vee S^1$ where $T = S^1 \times S^1$ is a 2-torus. Let E to be the covering space defined by the 2-fold covering of S^1 . Note that E is homeomorphic to the circle with two tori T attached at antipodal points. Thus, E is homotopy equivalent to $T \vee T \vee S^1$. By Theorem 3.6 and Lemma 2.7

$$TC(B) \leq TC^M(T) + TC^M(S^1) - 1 = \text{cat}(T) + \text{cat}(S^1) - 1 = 3 + 2 - 1 = 4.$$

On the other hand by Proposition 3.6,

$$TC(E) \geq \text{cat}((T \vee S^1) \times T) = 3 + 3 - 1 = 5.$$

(2) Consider $X = (S^3 \times S^3) \vee S^1$. Since $S^3 \times S^3$ is a connected Lie group, by Lemma 2.7, $TC^M(S^3 \times S^3) = \text{cat}(S^3 \times S^3) = 3$. By Theorem 3.6

$$TC(X) \leq TC^M(S^3 \times S^3) + TC^M(S^1) - 1 = 3 + 2 - 1 = 4.$$

Note that the universal cover \tilde{X} is homotopy equivalent to an infinite wedge $Y = \bigvee^{\infty} (S^3 \times S^3)$. Then Y admits a retraction onto $(S^3 \times S^3) \vee (S^3 \times S^3)$. By Proposition 3.2, Theorem 3.6, and the cup-length lower bound on cat ,

$$TC(\tilde{X}) \geq TC((S^3 \times S^3) \vee (S^3 \times S^3)) \geq \text{cat}(S^3 \times S^3 \times S^3 \times S^3) \geq 5.$$

□

4. TOPOLOGICAL COMPLEXITY, LS-CATEGORY, AND SCHWARTZ GENUS

We say a subset $A \subset X$ can be *rel ∞ contracted to infinity* if for every compact subset $F \subset X$ there is a larger compact set $F \subset C$ and a homotopy $h_t : A \rightarrow X$ with $h_0 = 1_A$, $h_1(A) \cap F = \emptyset$ and $h_t(a) = a$ for $a \in A \setminus C$.

4.1. Definition. We define the *rel ∞ category* $\infty\text{-cat}(X)$ of a locally compact space X as the minimal k such that there is a cover $X = V_1 \cup \dots \cup V_k$ by closed subsets where each V_i can be *rel ∞ contracted to infinity*.

4.2. Remark. It follows from the definition that for every locally compact space X ,

$$\text{cat}(\alpha X) \leq \infty\text{-cat}(X)$$

where αX is the one-point compactification.

4.3. Question. Does the equality $\text{cat}(\alpha X) = \infty\text{-cat}(X)$ hold for all locally finite complexes with tame ends?

We recall that X has a tame end if there is a compactum $C \subset X$ such that $X \setminus \text{Int}(C) \cong \partial C \times [0, 1)$.

In the case when αX is a closed manifold this question could be related to the difference between the category and the ball-category for manifolds. We recall that for a closed n -manifold M , $\text{ballcat}(M) \leq k$ is there is a cover of M by k closed topological n -dimensional balls.

4.4. Proposition. *For any closed n -manifold M and any $x_0 \in M$,*

$$\text{cat}(M) \leq \infty\text{-cat}(M \setminus \{x_0\}) \leq \text{ballcat}(M) \leq \text{cat}(M) + 1.$$

Proof. In view of Remark 4.3 and some known fact about the ball-category [CLOT], only the second inequality needs a proof. Let $\text{ballcat}(M) = m$ and let B_1, \dots, B_m be a cover of M by topological closed n -balls such that $x_0 \notin \partial B_i$ for all i . Then all $B_i \setminus \{x_0\}$ can be *rel ∞ contracted* in $M \setminus \{x_0\}$ to x_0 . □

Since the one-point compactification of $X \times X$ with the diagonal ΔX removed is the quotient space $(X \times X)/\Delta X$, the following theorem shows that Question 4.3 is closely related to characterization of the topological complexity TC^M by means of the LS-category.

4.5. Theorem. *For any compact ENR X ,*

$$\text{cat}((X \times X)/\Delta X) \leq TC^M(X) \leq \infty\text{-cat}((X \times X) \setminus \Delta X).$$

Proof. Suppose that $TC^M(X) = k$. Then by the definition there is an open cover U_1, \dots, U_k of $X \times X$ with continuous reserved sections $s_i : U_i \rightarrow PX$ of $\pi : PX \rightarrow X \times X$. By Proposition 3.1 there are strict deformations of U_i in $X \times X$ to the diagonal ΔX . They define the deformations of $U_i/(U_i \cap \Delta X)$ to the point $\{\Delta X\}$ in $(X \times X)/\Delta X$. Thus, $\text{cat}((X \times X)/\Delta X) \leq k$.

Let $\infty\text{-cat}((X \times X) \setminus \Delta X) = k$ and let $(X \times X) \setminus \Delta X = F_1 \cup \dots \cup F_k$ be the union of k closed sets *rel* ∞ contractible to infinity. Let W be a neighborhood of the diagonal ΔX in $X \times X$ that admits a deformation retraction r_t to ΔX . Let h_t^i be a deformation of F_i into W . Then the concatenation of h_t^i and r_t defines a deformation H_i of F_i to the diagonal. Let $\bar{F}_i = F_i \cup \Delta X$. Note that H_i together with identity on ΔX define a strict deformation of \bar{F}_i to the diagonal. \square

4.6. Remark. For the topological complexity $TC(X)$ a weaker version of the first inequality from Theorem 4.5 was proven in [F2], Lemma 18.3.

$$\text{cat}((X \times X)/\Delta X) - 1 \leq TC(X).$$

The topological complexity of X equals the Schwarz genus of a certain fibration. It turns out that for general fibrations we still have the inequalities similar to Theorem 4.5.

4.7. Theorem. *For any fibration of compact spaces $p : X \rightarrow Y$,*

$$\text{cat}(C_p) - 1 \leq \text{sg}(p) \leq \infty\text{-cat}(C_p \setminus \{*\}).$$

Proof. We claim that if a subset $U \subset Y$ admits a section $s : U \rightarrow X$, then U is contractible in C_p . Indeed, it can be moved to X in the mapping cylinder M_p . Since the cone $\text{Con}(X)$ is contained in C_p , it could be further contracted to a point. Moreover, the mapping cylinder $\hat{U} = M_{p|_{p^{-1}(U)}}$ of the restriction of p to the preimage $p^{-1}(U)$ is contractible in C_p , since it can be pushed to U first. If Y is covered by n open sets U_1, \dots, U_n each of which admits a section of p , then the mapping cylinder M_p can be covered by n sets $\hat{U}_1, \dots, \hat{U}_n$ all contractible in the mapping cylinder C_p . Since $C_p = M_p \cup \text{Con}(X)$, the open enlargements of the sets $\hat{U}_1, \dots, \hat{U}_n$, and $\text{Con}(X)$ define an open cover of C_p by $n + 1$ elements all contractible in C_p . Hence $\text{cat}(C_p) - 1 \leq \text{sg}(p)$.

Suppose that $\infty\text{-cat}(C_p \setminus \{*\}) \leq n$. Let V_1, \dots, V_n be a closed cover of $C_p \setminus \{*\}$ by sets that can be *rel* ∞ contracted to infinity. Let

$$H_i : V_i \times I \rightarrow C_p \setminus \{*\}$$

be a contraction such that

$$H_i(V_i \times 1) \subset \text{Con}(X) \setminus \{*\} \subset C_p \setminus \{*\}.$$

We define $F_i = V_i \cap Y \subset C_p$. Let $\pi : \text{Con}(X) \setminus \{*\} \rightarrow X$ be the projection. By the Homotopy Lifting Property, the homotopy $p \circ H_i|_{F_i \times [0,1]} : F_i \times [0,1] \rightarrow Y$ has a lift $H'_i : F_i \times [0,1] \rightarrow X$ which coincides with $\pi \circ H_i$ on $F_i \times 1$. Then H'_i restricted to $F_i \times 0$ is a section of p over F_i . Thus, $sg(p) \leq \infty\text{-cat}(C_p \setminus \{*\})$. \square

The following example shows that neither of the two inequalities of Theorem 4.7 can be improved.

4.8. Example. (1) For the identity map $1_X : X \rightarrow X$ in view of the equality $C_{1_X} = \text{Con}(X)$ we obtain:

$$\text{cat}(C_{1_X}) - 1 = 0 < sg(1_X) = 1 = \text{cat}(\text{Con}(X)) = \infty\text{-cat}(C_{1_X} \setminus \{*\}).$$

For the square map $p : S^1 \rightarrow S^1$, $p(z) = z^2$,

$$\text{cat}(C_p) - 1 = 2 = sg(p) < 3 = \text{cat}(C_p) \leq \infty\text{-cat}(C_p \setminus \{*\}),$$

since $C_p = \mathbb{R}P^2$ and $\text{cat}(\mathbb{R}P^2) = 3$.

5. ON THE ARNOLD-KUIPER THEOREM

5.1. Theorem. *The non-reduced Lusternik-Schnirelmann category of the orbit space $\mathbb{C}P^2/\mathbb{Z}_2$ of the action of \mathbb{Z}_2 on the complex projective plane $\mathbb{C}P^2$ by the conjugation is 2,*

$$\text{cat}(\mathbb{C}P^2/\mathbb{Z}_2) = 2.$$

5.2. Corollary (Arnold, Kuiper). *The orbit space $\mathbb{C}P^2/\mathbb{Z}_2$ of the action of \mathbb{Z}_2 on the complex projective plane $\mathbb{C}P^2$ by the conjugation is a 4-sphere.*

Proof. Clearly, the fixed point set of this action is a real projective plane

$$\begin{aligned} \mathbb{R}P^2 &= \{[a : b : c] \mid a, b, c \in \mathbb{R}, |a| + |b| + |c| \neq 0\} \subset \\ &\subset \{[a : b : c] \mid a, b, c \in \mathbb{C}, |a| + |b| + |c| \neq 0\} = \mathbb{C}P^2. \end{aligned}$$

Moreover, the action preserves the normal bundle to $\mathbb{R}P^2$. Therefore, the orbit space $\mathbb{C}P^2/\mathbb{Z}_2$ is a 4-manifold. A closed n -manifold of the category 2 is homotopy equivalent to the n -sphere (see [CLOT]). Then by Freedman's theorem [Fr], $\mathbb{C}P^2/\mathbb{Z}_2$ is homeomorphic to the 4-sphere. \square

5.3. Remark. We note that Arnold and Kuiper proved a diffeomorphism theorem. Since the smooth 4-dimensional Poincare conjecture is still a conjecture, here we can provide only a homeomorphism.

We identify the 2-sphere S^2 with the one-point compactification $\mathbb{C} \cup \infty$ of the complex plane. Then \mathbb{Z}_2 -action on \mathbb{C} by the conjugation extends to an action on S^2 . Clearly, a \mathbb{Z}_2 -action on S^2 extends to an action on the symmetric n th power $SP^n(S^2)$ of S^2 . We recall that $SP^n X = X^n / \Sigma_n$ is the orbit space on the n th power X^n under the action of the symmetric group Σ_n by permutation of coordinates.

5.4. Proposition. *There is a \mathbb{Z}_2 -equivariant homeomorphism between complex projective space \mathbb{CP}^2 and the symmetric square $SP^2(S^2)$.*

Proof. The points $[a : b : c] \in \mathbb{CP}^2$ are in bijection with non-degenerate quadratics $ax^2 + bxy + cy^2$. Any factorization of this quadratic

$$ax^2 + bxy + cy^2 = (a_1x + b_1y)(a_2x + b_2y)$$

defines the same non-ordered (perhaps repeated) pairs of points

$$\frac{a_1}{b_1}, \frac{a_2}{b_2} \in \mathbb{C} \cup \infty = S^2.$$

Note that the non-degeneration condition $|a| + |b| + |c| \neq 0$ implies that a_i and b_i cannot be all equal zero for $i = 1, 2$. Also we use the standard convention $\frac{z}{0} = \infty$ for any $z \in \mathbb{C}$.

This correspondence is the required homeomorphism. \square

5.5. Remark. The above proposition is an equivariant version of the well-known fact: $\mathbb{CP}^n \cong SP^n(S^2)$.

Proof of Theorem 5.1. We present $M = SP^2(S^2)/\mathbb{Z}_2 = F \cup U$ as a union of two contractible sets one closed and one open. Note that the set $U = SP^2(\mathbb{C})/\mathbb{Z}_2$ is open and contractible, since \mathbb{C} is contractible to a point equivariantly. The equator $S^1 = \mathbb{R} \cup \infty \subset S^2$ separates S^2 in two hemispheres D_- and D_+ . We show that the complement $F = M \setminus U$ admits a continuous bijection onto the closed upper hemisphere \bar{D}_+ . Indeed, it consists of non-ordered pairs of pairs $\{\infty, z\}$, $\{\infty, \bar{z}\}$ where $z \in \bar{D}_+$. This defines the bijection which is clearly continuous. Since F is compact, it is homeomorphic to \bar{D}_+ and hence is contractible. Since F is an absolute retract and M is absolute neighborhood retract, there is an open neighborhood V of F in M that contracts to F in M and, hence, to a point. Thus, M is covered by two open sets U and V , both contractible in M . \square

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